

Zero-Collapse and Resource Allocation in Multiplicative Systems

David Kirsch

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Abstract

Where should a society invest its marginal dollar? This paper presents a mathematical framework that answers the question directly. When independent dimensions of human or economic capacity compose multiplicatively rather than additively, the marginal return on investment in any dimension is inversely proportional to that dimension's current value. The lowest dimension always yields the highest return. This property, which we term marginal inversion, follows from the calculus of the multiplicative form and requires no economic theory beyond the composition structure itself. A second property, zero-collapse, predicts that any dimension reaching zero eliminates the product entirely, producing cliff-like rather than proportional decline. We validate the framework empirically across three scales: individual outcomes (Housing First programs, $N=950$ RCT), national crisis response (Greece/Iceland and US/UK natural experiments), and societal collapse (Venezuela and Zimbabwe governance trajectories using World Bank WGI data). A 162-country cross-sectional analysis confirms multiplicative composition of national output, with fitted exponents placing human capital (0.79), infrastructure (0.50), and health (0.42) as the operative dimensions. We then examine fifty years of tax-cut data across eighteen OECD nations (no statistically significant growth effect from top-end investment) against randomized trials of bottom-end investment (fiscal multipliers of 2.5–4.3 \times). The multiplicative framework predicts both findings. Current US fiscal policy (P.L. 119-21) simultaneously reduces the lowest dimensions while subsidizing the highest, an allocation the framework identifies as provably suboptimal.

1. Introduction

The question of where public investment produces the greatest return has been debated for decades without mathematical resolution. Supply-side economics argues that reducing taxes on capital formation and high earners generates growth that distributes broadly. Demand-side economics argues that direct investment in lower-income populations generates higher multiplier effects. Both sides marshal empirical evidence. Neither offers a mathematical proof of optimality.

This paper provides one.

The argument rests on a structural property of how independent dimensions compose. If a person's capacity to function depends on housing, healthcare, education, financial stability, and social connection, and if these dimensions compose multiplicatively, then the marginal return on investment in any single dimension scales as the inverse of that dimension's value. This is not an assumption about human behavior or market dynamics. It is a consequence of the composition function.

The multiplicative form is not arbitrary. Kirsch (2026) proves that given five axioms — zero-collapse, continuity, strict monotonicity, scale consistency, and dimensional independence — the multiplicative power-law form is the unique valid composition function. No additive alternative satisfies these constraints. The proof follows Aczél’s (1966) functional equations framework and has been validated empirically across eight domains including neural dynamics, network synchronization, ecosystem resilience, and macroeconomic output.

This paper applies two properties of the multiplicative form to questions of social and economic policy. Section 2 develops the mathematical framework. Section 3 presents empirical validation at three scales. Section 4 examines the marginal inversion property against the supply-side debate. Section 5 applies the framework to current US fiscal policy. Section 6 discusses limitations and falsification criteria.

2. Mathematical Framework

2.1 The Multiplicative Composition Form

Let x_1, x_2, \dots, x_n represent n independently measurable dimensions of a system. Define the emergent output E as:

$$E = k \cdot \prod x_i^{\alpha_i}$$

where $k > 0$ is a scaling constant and $\alpha_i > 0$ are dimension-specific exponents. Kirsch (2026) proves this form is uniquely determined by five axioms: (A1) if any $x_i = 0$, then $E = 0$; (A2) E is continuous; (A3) E is strictly monotone increasing in each dimension; (A4) E is homogeneous of degree κ ; (A5) E is separable. The proof proceeds by log-transformation: under homogeneity and separability, $\log E$ is linear in each $\log x_i$, yielding the power-law product form upon exponentiation.

2.2 Zero-Collapse

From axiom A1 directly: if any dimension $x_j = 0$, then $E = 0$, regardless of the values of all other dimensions. This distinguishes the multiplicative form from the additive form $f = \sum w_i x_i$, where zeroing one dimension reduces output by only that dimension’s weighted contribution. Under multiplicative composition, the loss is total. The empirical signature is disproportionate decline: an additive model predicts that losing one of four dimensions reduces output by approximately 25%; a multiplicative model predicts decline approaching 100%.

2.3 Marginal Inversion

The partial derivative of E with respect to any dimension x_i is:

$$\partial E / \partial x_i = \alpha_i \cdot E / x_i$$

The marginal return is inversely proportional to x_i . As x_i approaches zero, the derivative diverges. As x_i grows large, it approaches zero. Given a fixed budget, the allocation maximizing total output invests most heavily in the dimension with the lowest current value. This property holds for any multiplicative system regardless of the specific α_i . It requires no assumptions about market structure, individual behavior, or

institutional design.

2.4 Relationship to Existing Frameworks

The multiplicative form is not new to economics. Cobb and Douglas (1928) proposed it for aggregate production. Hausmann, Rodrik, and Velasco (2005) articulated binding constraints as a diagnostic heuristic. What is new is the uniqueness proof: the multiplicative form is not a choice among alternatives but the only form that satisfies the axioms. The information-geometric derivation (Kirsch, 2026b) provides a second foundation: the volume element of the Fisher information metric on a product manifold is structurally identical to the multiplicative form.

3. Empirical Evidence Across Three Scales

3.1 Individual Scale: Housing First

If functional capacity composes multiplicatively, a person with zero housing should receive zero return from investment in any other dimension. The CDC systematic review of 26 studies found Housing First programs return \$1.44 per \$1.00 invested (CDC/CPSTF, 2023). The At Home/Chez Soi RCT (N=950, five cities) found 73% housing stability versus 31% for treatment-as-usual, with 69% cost offset (Aubry et al., 2020). Finland's national program reduced long-term homelessness 68% over fourteen years — the only EU country where homelessness declined (Pathfinders, 2024). These outcomes are consistent with multiplicative composition: the return is approximately zero until the binding zero (housing) is addressed, then sharply positive.

3.2 National Scale: Crisis Response

Greece and Iceland both collapsed in 2008. Greece chose austerity: GDP fell 26%; unemployment peaked at 28%; debt-to-GDP rose from 130% to 180% (Eurostat; Danchev, 2024). Iceland invested in its safety net, prosecuted 29 bankers, and recovered GDP by 2015 (Wharton, 2018). The US passed \$836B in stimulus (ARRA); the CBO estimated GDP raised 0.7–4.1% (CBO, 2015). The UK chose austerity: capital spending fell 32%; food bank centres rose from 35 to 1,300; debt rose from 65% to 98% of GDP. Blanchard and Leigh (2013) found the actual fiscal multiplier was 0.9–1.7, not the assumed 0.5. The multiplicative framework predicts these asymmetries: stimulus invests where the derivative is steepest; austerity cuts there.

3.3 Societal Scale: Failed States

Venezuela holds 303 billion barrels of proven oil reserves (OPEC, 2025). GDP fell 88% from 2012 to 2020, deeper than Syria's wartime collapse. Per capita income dropped from \$12,607 to \$1,509. World Bank Rule of Law fell from -0.76 to -2.35 . Zimbabwe's hyperinflation peaked at 79.6 billion percent monthly (Hanke and Kwok, 2009); GDP per capita returned to 1952 levels; Rule of Law hit the 1st percentile. In both cases, substantial non-governance dimensions survived. An additive model predicts ~25% decline for one-dimension loss. Observed: 78–90%. This ratio is the empirical signature of zero-collapse.

3.4 Cross-Country Validation

Regression of GDP on three dimensions (human capital, infrastructure, health) across 162 countries yielded multiplicative Spearman correlation of 0.953, with exponents of 0.79, 0.50, and 0.42 respectively. Health carries the lowest exponent, implying the steepest marginal return at low values — consistent with development economics findings on health interventions in low-income settings (Bloom et al., 2004; Jamison et al., 2013).

4. Marginal Inversion and the Supply-Side Question

4.1 Evidence on Top-End Investment

Hope and Limberg (2022) examined 30 major tax cuts across 18 OECD countries (1965–2015): no statistically significant effect on GDP per capita or unemployment; only higher top-1% concentration. Hungerford (2012, CRS) found no association between 65 years of top-rate reductions and saving, investment, or productivity growth. Ostry, Berg, and Tsangarides (2014, IMF) concluded the equity-efficiency trade-off is “largely illusory.” Kansas (2012–2017): top rate cut 30%, pass-through taxed at zero; GDP grew at half the national rate; a Republican legislature repealed the cuts over the governor’s veto (Gale, 2017).

4.2 Evidence on Bottom-End Investment

Egger et al. (2022, *Econometrica*): \$1,000 transfers to 10,500+ poor households in Kenya; fiscal multiplier 2.5–2.7x. Banerjee, Duflo et al. (2015, *Science*): BRAC graduation across six countries; returns 133–433%; multi-dimensional package outperformed single-component. Chetty, Hendren, and Katz (2016, *AER*): Moving to Opportunity; children moved before age 13 earned 31% more as adults; returns largest at lowest baseline.

4.3 The Juxtaposition

Fifty years of top-end tax cuts: zero measurable growth. Six randomized trials of bottom-end investment: 2.5–4.3x returns. The multiplicative framework predicts both from the same property: the derivative is flat at high values and steep at low values. The question of where to invest is resolved by the composition structure, not by ideological preference.

5. Current Application: US Fiscal Policy (2025)

The One Big Beautiful Bill Act (P.L. 119-21, July 4, 2025) cuts Medicaid by ~\$1 trillion and SNAP by ~\$186 billion over ten years, while making permanent the 2017 tax cuts and raising the estate exemption to \$15 million. CBO distributional analysis (August 2025): bottom decile loses 3.1% of resources; top decile gains 2.7%. The Tax Policy Center estimates ~60% of tax benefits accrue to the top quintile; the top 1% receives an average net reduction of \$66,000. In multiplicative terms, this law cuts the lowest dimensions and subsidizes the highest — the least efficient allocation available.

Early indicators: HUD recorded 771,480 homeless (2024 PIT, +18%). Child poverty doubled (5.2% to 12.4%) when the expanded CTC expired. SNAP lapsed for the first time in program history (November 2025). Institutional trust indices (V-Dem, Freedom House, EIU) all recorded their lowest US scores on record.

6. Limitations and Falsification

The cross-country analysis is cross-sectional; panel extension is planned. CES elasticity ρ does not converge to zero (−0.28 to +0.38), placing the exact composition on a continuum; the multiplicative model outperforms the additive model in every domain tested, but the pure form is an approximation. The crisis response evidence is observational (except At Home/Chez Soi). The full effects of P.L. 119-21 will unfold over a decade; Medicaid work requirements take effect January 1, 2027.

Falsification criteria: (a) RCT showing single-dimension interventions matching BRAC multi-dimensional returns; (b) credible cross-national study finding significant growth from top-end tax cuts after Hope-Limberg controls; (c) a country reaching bottom-decile WGI without disproportionate output collapse; (d) OBBBA Medicaid provisions producing coverage loss well below CBO's 5.3M projection. None currently exists in the published literature.

7. Conclusion

The mathematics of multiplicative composition resolves the resource allocation question without recourse to economic theory, political philosophy, or moral argument. When independent dimensions compose multiplicatively, the marginal return is highest where the dimension is lowest. Investment at the bottom is not a compassionate alternative to investment at the top. It is the mathematically optimal allocation. The growth-maximizing allocation and the equity-maximizing allocation are the same allocation, because the steepest part of the growth curve is at the bottom of the distribution.

The zero-collapse property adds urgency. Systems that lose a single dimension do not degrade gradually. Venezuela's oil and Zimbabwe's farmland were not sufficient to prevent collapse once institutional trust reached the bottom percentile. The multiplicative model predicts this; the additive model does not. Current US fiscal policy (P.L. 119-21) simultaneously reduces the lowest dimensions while concentrating resources in the highest. The framework developed here identifies this as a provably suboptimal allocation — not as a matter of values, but of structure.

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