Emergence as Information Volume: The Geometric Foundation of Multiplicative Composition

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Abstract

A companion paper proves that emergent properties—outputs requiring the joint satisfaction of independent, non-substitutable preconditions—compose via the unique multiplicative power-law $f(\mathbf{x}) = k \prod x_i^{\alpha_i}$. That proof is axiomatic: five structural postulates determine the functional form. This paper provides the geometric foundation. We show that the multiplicative law arises naturally from the information geometry of independent observation channels. On a product statistical manifold $\mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_N$ equipped with the Fisher-Rao metric g, the Riemannian volume density factorizes as $\sqrt{\det g} = \prod_i \sqrt{g_{ii}(\theta_i)}$. We construct a concrete statistical model—N independent channels with power-law response functions $\mu_i(\theta_i) = \theta_i^{q_i}$, $q_i > 1$—under which $\sqrt{\det g(\boldsymbol{\theta})} = k \prod \theta_i^{\alpha_i}$ with $\alpha_i = q_i - 1 > 0$, recovering the multiplicative composition law exactly. The five axioms of the companion paper are shown to be consequences of the geometry. We identify a duality between scalar curvature (additive under products) and volume density (multiplicative under products), giving geometric content to the scope condition: substitutable dimensions compose through curvature; non-substitutable dimensions compose through volume.

Keywords: information geometry, Fisher information, Riemannian volume element, multiplicative composition, statistical manifold, emergence

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1 Introduction

In the companion paper, we proved that the composition of independent, non-substitutable dimensions producing an emergent property is uniquely determined by five structural axioms to be the multiplicative power-law form:

$$f(x_1, \ldots, x_N) = k \prod_{i=1}^{N} x_i^{\alpha_i}, \quad k > 0,\ \alpha_i > 0. \tag{1}$$

This paper answers the question: *why* these axioms? The axioms are consequences of the information geometry of independent observation channels. The multiplicative form is the Riemannian volume density on a product statistical manifold equipped with the Fisher-Rao metric, under a natural class of statistical models.

The argument proceeds in three layers. *Layer 1*: On any product Riemannian manifold, the volume density factorizes multiplicatively (standard Riemannian geometry). *Layer 2*: Under power-law response channels, this factorization yields the power-law form exactly, with exponents $\alpha_i = q_i - 1$ determined by the response order. *Layer 3*: The five axioms are recovered as geometric consequences.

A second contribution is the curvature-volume duality: scalar curvature is additive under Cartesian products, while volume density is multiplicative. This gives geometric content to the scope condition: substitutable dimensions compose through curvature (additive); non-substitutable dimensions compose through volume (multiplicative).

2 Statistical Manifolds and the Fisher-Rao Metric

A *statistical manifold* is a smooth manifold whose points are probability distributions. The Fisher information matrix at θ is:

$$g_{ij}(\theta) = \mathrm{E}_\theta\left[\frac{\partial \log p(X \mid \theta)}{\partial \theta_i} \cdot \frac{\partial \log p(X \mid \theta)}{\partial \theta_j}\right]. \tag{2}$$

By Chentsov's theorem, the Fisher-Rao metric is the unique Riemannian metric invariant under sufficient statistics. The Riemannian volume element is $d\mathcal{V} = \sqrt{\det g(\theta)}\, d\theta_1 \wedge \cdots \wedge d\theta_N$, where $\sqrt{\det g(\theta)}$ is the *informational volume density*—the number of statistically distinguishable distributions per unit parameter volume.

When parameters are independent ($p(x \mid \theta) = \prod p_i(x_i \mid \theta_i)$), the Fisher matrix is diagonal: $g_{ij} = g_{ii}(\theta_i)\delta_{ij}$.

3 Volume Factorization

Theorem 3.1 (Volume factorization). *Let $\mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_N$ be a product statistical manifold with diagonal Fisher metric. Then $\sqrt{\det g(\theta)} = \prod_{i=1}^{N} \sqrt{g_{ii}(\theta_i)}$.*

Proof. $\det g = \prod g_{ii}(\theta_i)$ for a diagonal matrix. Take the square root. □

The informational volume density is multiplicative. If we define $h_i(\theta_i) := \sqrt{g_{ii}(\theta_i)}$ as the *informational resolution* of dimension i, then $\sqrt{\det g} = \prod h_i(\theta_i)$.

4 The Power-Law Response Channel

Consider N independent channels with power-law response: $Y_i \sim p_i((y_i - \theta_i^{q_i})/\sigma_i)$, $q_i > 1$.

Proposition 4.1. *The Fisher information is* $g_{ii}(\theta_i) = (I_{p_i} q_i^2/\sigma_i^2)\theta_i^{2(q_i-1)}$.

Setting $\alpha_i = q_i - 1$ and applying the volume factorization:

$$\sqrt{\det g(\boldsymbol{\theta})} = k \prod_{i=1}^{N} \theta_i^{\alpha_i} \tag{3}$$

recovering the multiplicative composition law exactly. The exponent α_i is the elasticity of informational resolution with respect to the dimension value: $\alpha_i = d \log h_i / d \log \theta_i$.

5 Axiom Recovery

Under the power-law response model, the five axioms are geometric consequences:

1. **Zero-collapse.** At $\theta_i = 0$, Fisher information vanishes ($g_{ii}(0) = 0$), so $E(\boldsymbol{\theta}) = 0$.
2. **Continuity.** The Fisher metric is smooth on the interior.
3. **Monotonicity.** $\partial E/\partial \theta_i > 0$ since $\alpha_i > 0$.
4. **Homogeneity.** $E(\lambda\boldsymbol{\theta}) = \lambda^{\sum \alpha_i} E(\boldsymbol{\theta})$ from the power-law structure.
5. **Separability.** $E = \prod h_i$ from the product structure and diagonal metric.

6 The Curvature-Volume Duality

On a product manifold $\mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_N$:

Scalar curvature is additive: $R(\mathcal{M}_1 \times \mathcal{M}_2) = R(\mathcal{M}_1) + R(\mathcal{M}_2)$.

Volume density is multiplicative: $\sqrt{\det g} = \prod \sqrt{\det g_i}$.

This duality gives geometric content to the scope condition. Volume measures total distinguishable structure—zero along any axis means zero total (non-substitutable). Curvature measures local geometric complexity—other dimensions compensate when one degenerates (substitutable). The same manifold supports both invariants. The physical question determines which is relevant.

7 Discussion

The honest summary: geometry gives you multiplication; the specific model gives you the exponents. Layer 1 (volume factorization) is unconditional geometry. Layer 2 (power-law Fisher information) is model-specific. Layer 3 (exponent values) is empirical. The power-law response is not the only model with vanishing Fisher information at zero, but it is distinguished by scale invariance (producing the homogeneity axiom). Relaxing it opens the CES family.

8 Conclusion

The multiplicative composition law has a geometric origin: it is the Riemannian volume density on a product statistical manifold with the Fisher-Rao metric. The five axioms are algebraic shadows of the geometry. The curvature-volume duality gives the scope condition geometric content. The multiplicative structure has been independently discovered nine times across two centuries. This paper suggests a reason: it is the geometry of information.

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